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LETTER TO THE EDITOR

An order parameter for networks of automata

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Abstract. An exact polynomial equation is given for the size of the stable core of networks of automata with random connections. When the connectivity K of a network equals 1, 2, 3, 4 or 5 this equation is exactly solvable. It is found that the size of the stable core is an order parameter for a phase transition well known from Kauffman's model. A new derivation of critical parameter values follows. The phase structure is found to be independent of the updating scheme used in the dynamical law for the network.

The dynamics of networks of formal neurons, spin glasses at low temperature, and other networks of automata have received much attention recently. The simplest networks consist of N binary variables $\sigma_i = 0, 1$, which evolve in discrete time according to

$$\sigma_i(t+1) = f_i(\sigma_{j_1(i)}(t), \sigma_{j_2(i)}(t), \ldots, \sigma_{j_K(i)}(t)). \tag{1}$$

Different types of networks are distinguished by the way the functions f_i and the connections $j_1(i), \ldots, j_K(i)$ are chosen. For neural networks f_i is typically a threshold function of a linear combination of its arguments with the threshold and the weights of the linear combination carrying the memory. For the Hopfield model $K = N$ [1, 2], but so-called diluted models with $K \ll N$ and randomly chosen connections $j_1(i), \ldots, j_K(i)$ have also been studied with success [3–5]. In Ising spin glass relaxation dynamics f_i is again a threshold function of a linear combination of its arguments. The linear weights are random numbers, usually with a symmetry restriction. $K = N$ for the Sherrington–Kirkpatrick model [6] and $K \ll N$ for diluted models [7]. In Kauffman's genetic model for metabolic stability and cell differentiation in embryonic development $K \ll N$, and for each value of i the function f_i is chosen at random, as are the connections $j_1(i), \ldots, j_K(i)$ [8–10].

Diluted and randomly connected networks are collectively characterised by finite connectivity $K \ll N$ and randomly chosen connections $(j_1(i), \ldots, j_K(i))$, $i = 1, \ldots, N$. All such networks are to some extent solvable in the thermodynamical limit $N \to \infty$ when K is kept fixed at a finite value [3–5, 7, 11–15]. In this letter we show how one can compute the size of the stable core† of such networks. We give an exact equation for the transient time evolution of the size of the stable core, and we demonstrate that the size of the stable core at time infinity is an order parameter for a second-order transition between phases characterised by either 'chaotic' or 'frozen' dynamics of the network. This phase transition has previously been determined for Kauffman's model from equations for the time evolution of overlaps between pairs of configurations [11–13].

† Defined below.

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The stable core is the set of variables σ_i which develop in time to a constant value that is independent of the initial configuration [16, 17]. We denote its relative size by s, i.e. its absolute size is Ns.

In order to discuss how the stable core acquires its size, we introduce 'the stable core at time t', meaning those variables σ_i which at time t have attained 'stable' values. A 'stable' value is a value that remains constant for all later times *and* is independent of the initial configuration.

Let $s(t)$ denote the relative size of the stable core at time t. Then it is clear that $s(t)$ is non-decreasing. It is also clear that $s(1) = p_K$; any variable σ_i updated with a constant function f_i has reached its stable value after one update, no matter what the initial configuration was, and this is the case for only such variables. At any time $t+1$ there are $K+1$ mutually exclusive reasons that a variable σ_i may have attained its stable value.

(0) σ_i is updated with a function f_i which depends on variables $(\sigma_{j_1(i)}, \ldots, \sigma_{j_K(i)})$ that all are in the stable core at time t. Since $j_K(i)$ was chosen at random, this situation occurs with probability $s(t)^K$.

(1) σ_i is updated with a function f_i which depends on variables, of which all but one are in the stable core at time t. f_i happens to be a function which is *independent* of its one variable outside the stable core, when its $K-1$ variables *in* the core have their 'stable' values. This situation occurs with probability

$$Ks(t)^{K-1}(1-s(t))p_1$$

where p_1 is the probability that a function f_i in the network for given values of $K-1$ of its variables is independent of its Kth variable.

$\vdots$

(k) σ_i is updated with a function f_i which depends on variables, of which all but k are in the stable core at time t. f_i happens to be a function which is *independent* of its k variables outside the stable core, when its $K-k$ variables *in* the core have their 'stable' values. This situation occurs with probability

$$\binom{K}{k} s(t)^{K-k}(1-s(t))^k p_k$$

where p_k is the probability that a function f_i in the network for given values of $K-k$ of its variables is independent of its other k variables.

$\vdots$

(K) etc.

Since the relative size $s(t+1)$ of the stable core at time $t+1$ is also the probability that a variable σ_i belongs to it at that time, we can sum the probabilities above to an equation that gives $s(t+1)$ as a function of $s(t)$:

$$s(t+1) = P(s(t)) \equiv \sum_{k=0}^{K} \binom{K}{k} s(t)^{K-k}(1-s(t))^k p_k \tag{2}$$

where $p_0 = 1$. Equation (2) and its consequences are the main result of this letter.

$s(t)$ increases with time according to (2) until the limit value s is reached. s is found by setting $s(t+1) = s(t) = s$ in (2):

$$s = P(s). \tag{3}$$

Equation (3) is a stationarity condition for the time evolution of $s(t)$. It may also be read as a self-consistency condition on the size of the stable core at time infinity. When read this way, the $K+1$ reasons given above may be illustrated as in figure 1.

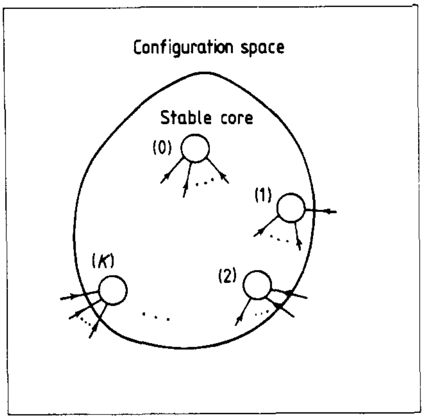

Figure 1. Symbolic illustration of (3).

Clearly $s=1$ solves (3) for any values of $p_1, \ldots, p_K$. For $K>1$ division of (3) by $1-s$ gives

$$s+s^2+\ldots+s^{K-1}=\sum_{k=1}^{K}\binom{K}{k}s^{K-k}(1-s)^{k-1}p_k. \tag{4}$$

Equation (4) is also solved by $s=1$ provided

$$p_1=1-1/K. \tag{5}$$

We shall soon see that this is a critical condition for the network.

For $K=2$, (4) is linear in s and is solved by

$$s=\frac{p_2}{1-2p_1+p_2}. \tag{6}$$

This solution is less than or equal to 1 for p_1 less than or equal to $\frac{1}{2}$. For $p_1=\frac{1}{2}-\varepsilon$, equation (5) gives $s=1-\mathrm{O}(\varepsilon)$, i.e. $p_1=\frac{1}{2}$ is a critical value with critical exponent 1 for $s(p_1)$.

For $K=3$, (4) is a quadratic equation in s and is solved by

$$s=\frac{1-3p_2+2p_3\pm[(1-3p_2)^2+4p_3(2-3p_1)]^{1/2}}{2(-1+3p_1-3p_2+p_3)}. \tag{7}$$

For $p_1=\frac{2}{3}$, (5) gives $s=1$. The critical exponent for $s(p_1)$ is 1, except when $p_2=\frac{1}{3}$; then the exponent is $\frac{1}{2}$, assuming $p_3>0$.

For $K=4$ and for $K=5$, (4) is a cubic or quartic equation in s, and therefore may be solved exactly for s in these cases also. We do not give the results here, because they are rather complicated expressions, which we shall not need.

For any positive integer value of K we have $P(0)=p_K$ and $P'(1)=K(1-p_1)$. Hence, if $p_K>0$ and $p_1<1-1/K$, there is a second solution to (3) in the interval $[p_K, 1]$ besides the solution $s=1$ (see figure 2). When there is more than one solution to (3) in the interval $[0, 1]$, we must return to (2) to pick out the relevant solution.

The evolution described by (2) is an iterated map. Figure 2 shows that the smallest of the solutions to the corresponding fixed-point equation (3) is attractive, and hence the relevant one. The two solutions shown change roles at the value for p_1 given in (5), which is therefore the critical condition. The last statement is of course valid only

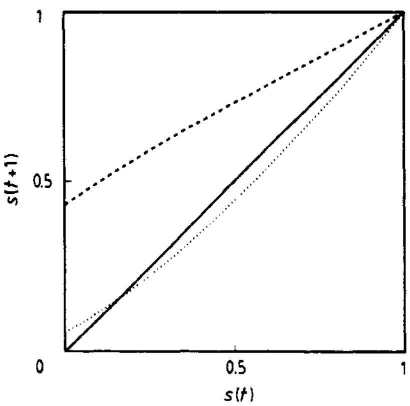

Figure 2. Graphical representation of (2) and (3). The dotted curve is a typical graph for $P(s)$ for $p_1 < 1-1/K$. The broken curve is a typical graph for $P(s)$ for $p_1 > 1-1/K$. The curves are for the case $K=3$: $p=0.7$ (dotted) and $p=0.9$ (broken).

when $P(s)$ behaves qualitatively as shown in figure 2, i.e. (3) has at most two solutions in the interval [0, 1], and $P''(1) > 0$ when (5) is satisfied. In that case (5) is the locus in parameter space of a second-order phase transition. Figure 2 is based upon the following example.

Example. Choose the functions f_i randomly among all Boolean functions, but with a bias such that $f_i(\sigma_1, \ldots, \sigma_K) = 1$ with probability p and $f_i(\sigma_1, \ldots, \sigma_K) = 0$ with probability $1-p$ independently for each of the 2^K possible input configurations $(\sigma_1, \ldots, \sigma_K)$. This is a variant of Kauffman's genetic model for metabolic stability and cell differentiation in embryonic development [8–10]. It gives $p_k = p^{2^k} + (1-p)^{2^k}$ and the critical value for p is

$$p_{\mathrm{crit}} = \tfrac{1}{2}[1 \pm (1-2/K)^{1/2}] \qquad \text{for} \qquad K \geq 2. \tag{8}$$

The symmetry with respect to $p = \frac{1}{2}$ in (7) is due to the equivalence between the values 0 and 1 for σ.

A subtlety: p_k was defined essentially as the probability that a function of k variables is a constant function. While (2) is valid for any value of N, the value for p_k given above is correct only if the k variables are independent. This is the case for $N = \infty$, but not for finite N. A simple example is provided by the case $k = 2$: the function XOR is constant, if its two arguments happen to be the same variable, thus contributing to p_2 with a term of $O(1/N)$, the probability that two randomly chosen inputs to XOR are identical.

Figure 3 shows the size s of the stable core as a function of p in the case $K=3$. We see two phases: the frozen phase characterised by $s = 1$ for $p > p_{\mathrm{crit}} = 0.788\,67\ldots$ and the chaotic phase characterised by $s < 1$ for $p < p_{\mathrm{crit}}$. The name for the frozen phase is self-explanatory: almost all spins are in the stable core. Whatever the initial configuration was, after a finite time these spins become constant; they freeze. In the chaotic phase a finite fraction of all spins are not in the stable core. After a finite time the evolution of a spin configuration is cyclic with a period that grows with N as $O(\exp(\alpha(K)N))$ [8–10]. Consequently, for $N \to \infty$ the limit behaviour is not cyclic in time, but chaotic.

Another example. The case $K = 1$ is shown to be exactly solvable and studied extensively in [13, 14].

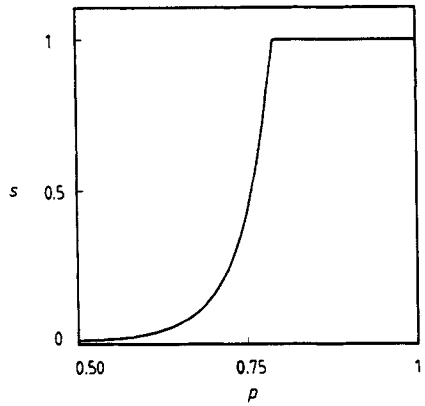

Figure 3. The size of the stable core as a function of p in the case $K=3$.

For given connectivity K each function f_i can be chosen in 2^{2^K} different ways. In the most general situation the choice is characterised by $2^{2^K}-1$ parameters, deriving from the 2^{2^K} probabilities that a function f_i is chosen equal to each of the 2^{2^K} functions possible. These 2^{2^K} probabilities sum to 1, leaving $2^{2^K}-1$ free parameters. The $(2^{2^K}-1)$-dimensional parameter space is partitioned in two regions by a surface given by (5). Parameter values in the region having $p_1 > 1-1/K$ result in frozen behaviour in the network. The other region corresponds to chaotic behaviour.

The time evolution in (2) is a result of the 'synchronous' or 'parallel' updating of the variables chosen with (1), but the self-consistency equation (3) for the relative size of the stable core at time infinity is independent of the particular time evolution chosen with (1). Had we instead chosen 'sequential updating', or 'random sequential updating', or any other algorithm that sooner or later will update any variable in the network, then (3) would still be valid and consequently so would the phase structure just found. Thus, the phase structure found above is independent of the updating scheme.

We have already seen that the phase structure found above is independent of the values of $p_2, p_3, \ldots, p_K$ as long as (3) has at most two solutions in the interval [0, 1], and $P''(1) > 0$ when (5) is satisfied.

We conclude that the phase structure of a randomly connected network of automata is essentially determined by the values of p_1 and K.

The recursive reasoning applied here to the size of the stable core has also been applied to the probability distribution for the average value of spins outside the stable core [15].

I have benefited greatly from discussions with Bernard Derrida.

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